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REACTION OF A PIEZOCERAMIC SHELL TO CONCENTRATED DYNAMICAL ACTIONS*

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A solution of the equations of motion of an infinite, cylindrical piezoceramic shell, the facial surfaces of which are not covered by electrodes but subjected to a periodic system of concentrated forces varying harmonically with time, is constructed. Green's function method is used for this purpose. In the case of regular roots of the dispersion equation the unique solution is picked out on the basis of limiting absorption principle. The irregular roots determine the spectrum of resonance frequencies. An analytical and numerical analysis of the roots of the dispersion equation is carried out. A qualitative picture of the wave process is given. The results of a calculation of the amplitude-frequency characteristics of the displacement and the electric field potential are presented as well as a comparison with a non-electric shell. The free vibrations of piezoceramic shells are considered in /1, 2/ and the forced vibrations of such shells in /3, 4/.

1. Let us consider a cylindrical piezoceramic shell which is referred to the orthogonal coordinates α, β and z, polarized along the α coordinate and loaded with a system of concentrated forces which are periodic with respect to β and vary harmonically with time. The facial surfaces of the shell are free from electrodes and are bounded by a vacuum. When account is taken of the equations of state /5/, the equations for the steady-state vibrations of such a shell have the form

$$L_{ij}u_j = P_i\delta(\alpha, \beta) + \delta_i\rho h\omega^2 u_i \qquad (1.1)$$
$$\delta_1 = \delta_2 = -1,\ \delta_3 = 1,\ \delta_4 = 0,\ i, j = 1, 2, 3, 4$$

Here, $u_j(\alpha, \beta)$ are the amplitudes of the displacements $(j = 1, 2, 3)$, $u_4 = \varphi(\alpha, \beta)$ is the electric field potential in the shell, $\delta(\alpha, \beta)$ is a two-dimensional Dirac function, ω is the frequency, ρ and h are the density of the material and the thickness of the shell and P_i is the amplitude of the corresponding concentrated force.

The differential operators L_{ij} are written out in /6/ and the coefficients occurring in them have the form

$$c_{33} = c_{33}^E[1 - (c_{13}^E)^2/(c_{11}^E c_{33}^E)]$$
$$c_{13} = c_{13}^E[1 - c_{13}^E/c_{11}^E],\quad c_{11} = c_{11}^E[1 - (c_{12}^E/c_{11}^E)^2]$$
$$e_{31} = e_{31}^*[1 - c_{12}^E/c_{11}^E],\ e_{15} = e_{15}^*$$
$$e_{33} = e_{33}^*[1 - e_{31}^* c_{13}^E/(e_{33}^* c_{11}^E)],\quad c_{44} = c_{44}^E$$
$$\varepsilon_{33} = \varepsilon_{33}^s[1 + (e_{31}^*)^2/(c_{11}^E\varepsilon_{33}^s)],\ \varepsilon_{11} = \varepsilon_{11}^s$$

Here, c_{ij}^E are the coefficients of elasticity of the piezoceramic when the electric field is zero, ε_{11}^s and ε_{33}^s are the permittivities when the stresses are zero and e_{ij}^* are

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the piezoelectric moduli.

Apart from the mechanical Kirchhoff-Love hypothesis, it has been assumed when deriving Eqs.(1.1) that the electrical boundary conditions on the facial surfaces have the form $D_z = 0$ (D_z is the corresponding component of the electric induction vector).

By introducing representations of the displacements in terms of the resolvent functions ψ_i /7/, we arrive at a single tenth-order differential equation

$$L(\partial_1, \partial_2)E(\alpha, \beta) = \delta(\alpha, \beta) \tag{1.2}$$

$$\Psi_i = -P_i E(\alpha, \beta)/(F_1 R_2^2 a_5), \quad \partial_1 = \partial/\partial\alpha, \quad \partial_2 = \partial/\partial\beta$$

$$L(\partial_1, \partial_2) = \sum_{j=0}^{5} \frac{a_j}{a_5} \partial_1^{10-2j}\partial_2^{2j} + \lambda^2 \Big[\sum_{j=0}^{4} \frac{d_j^{(8)}}{a_5} \partial_1^{8-2j}\partial_2^{2j} +$$

$$\sum_{j=0}^{3} \frac{a_j' + d_j^{(6)}}{a_5} \partial_1^{6-2j}\partial_2^{2j} + \sum_{j=0}^{2} \frac{d_j^{(4)}}{a_5} \partial_1^{4-2j}\partial_2^{2j} + \sum_{j=0}^{1} \frac{d_j^{(2)}}{a_5} \partial_1^{2-2j}\partial_2^{2j} \Big]$$

The coefficients a_j, a_j' and F_1 are defined in /6/, $\lambda = R_2/h$, the quantities $d_j^{(i)}$ depend on the material, the shell **parameters** and the frequency of the exciting force and are expressed in terms of a_j and a_j' (which are not written out here in view of **their** complexity) and R_2 is the radius of the median surface of the cylindrical shell.

We will seek a fundamental solution in the form of a Fourier series in the coordinate β with a subsequent integral Fourier transform with respect to the variable α. In accordance with this, $E(\alpha, \beta)$ can be represented as follows:

$$E(\alpha, \beta) = \sum_{k=-\infty}^{\infty} C_k(\alpha)\, e^{ikn\beta} \tag{1.3}$$

$$C_k(\alpha) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(iknz\alpha)\, dz}{(kn)^9 \Delta_k(\gamma, z)}$$

The characteristic polynomial of the Fourier transform has the form

$$\Delta_k(\gamma, z) = \sum_{j=1}^{5} b_{10-2j}^{(k)} z^{10-2i}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.4}$$

The coefficients $b_i^{(k)}$ depend on a_j, a_j' and $d_j^{(i)}$, the shell parameters and the frequency parameter $\gamma^2 = \rho\omega^2 R_2^2/c_{11}$.

A numerical analysis of the characteristic polynomial (1.4) was carried out in order to evaluate the inversion integral. This enabled us to pick out a discrete series of values of the frequency parameter γ, for which the dispersion equation $\Delta_k(\gamma, z) = 0$ has multiple roots z. The critical values of the frequency parameter in the case of a shell made of PTZ-5 piezoceramic when $\lambda = 30$ and $n = 1$ (n is the number of concentrated actions) where: 0.007, 0.034, 0.082, 0.149, 0.236, 0.34, 0.467, 0.61, 0.733, 0.775, 0.957, 1.06, 1.16, 1.381, 1.41, 1.465, 1.621, 1.881, 2.16, 2.199, 2.236, 2.458, 2.93, 3.113, 3.16,... .

The characteristic behaviour of certain roots of the polynomials $\Delta_0(\gamma, z)$ and $\Delta_3(\gamma, z)$ as a function of γ is shown in Figs.1 and 2. It can be seen that, in the static case $(\gamma = 0)$ all of the roots are complex and, as γ increases (γ increases in the direction of the arrows on the curves), they approach the imaginary axis and appear as two multiple purely imaginary roots. These roots subsequently separate and two of them proceed towards one another until a multiple zero root appears while two others go in the opposite direction along the imaginary axis. The multiple zero root splits into two real roots which move in opposite directions along the real axis. Analogous pictures also hold when $k = 1, 2$. When $k = 3$, characteristic loops arise, leading to the appearance of just a single pair of multiple purely imaginary roots. It can be seen from the analysis that multiple roots of z appear at those values of the frequency parameter for which the branches of the roots of the polynomial lie in the purely imaginary or real plane.

The critical values of the frequency parameter divide up the continuous spectrum into intervals in which all of the roots z of the characteristic polynomial of the Fourier transform are simple and for which the inversion (1.3) exists.

In order to pick out the unique solution in the intervals of continuity it is necessary to specify the conditions which characterize the behaviour of the solution at infinity. In the given problem we shall make use of the principle of limiting absorption /8/. An analysis of the roots of the dispersion Eq.(1.4) with absorption shows that the negative real roots correspond to the limit points of a sequence of complex roots located in the third quadrant of the complex plane z while the positive roots correspond to the limit points of the sequence of roots in the first quadrant (Fig.3). The integration contour for calculating the functions

$C_k(\alpha)$ in (1.3) is selected in accordance with this. Physically, the procedure which has been indicated implies the construction of the fundamental solution which ensures the transport of energy from the source of the perturbation to infinity ($\alpha = \pm\infty$).

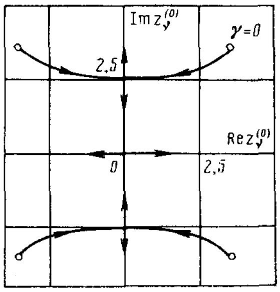

Fig.1

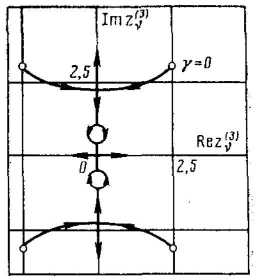

Fig.2

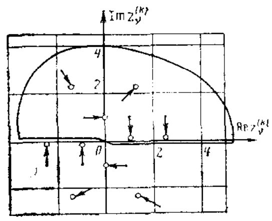

Fig.3

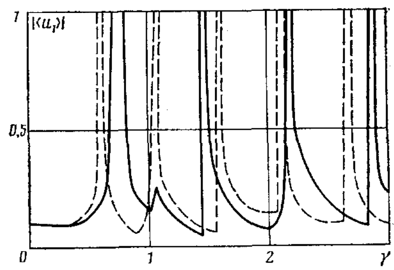

Fig.4

Hence, the fundamental solution in the intervals of continuity has the form

$$E(\alpha,\beta) = \frac{1}{T}\left\{\sum_{k=1}^{\infty}\left[2\cos(kn\beta)\sum_{\nu=1}^{\nu_0}\frac{\exp(iknx_\nu^{(k)}\alpha)}{(ikn)^9\Delta_k'(\gamma, x_\nu^{(k)})} + 2\,\mathrm{Re}\sum_{\nu=\nu_0+1}^{5}\frac{\exp[ikn(z_\nu^{(k)}\alpha+\beta)\,\mathrm{sgn}\,\alpha]}{(ikn)^9\Delta_k'(\gamma, z_\nu^{(k)})}\right] + C_0(\alpha)\right\}$$

$$C_0(\alpha) = \mathrm{sgn}\,\alpha\left[\frac{c_1|\alpha|}{2} + i\sum_{\nu=1}^{4}\frac{\exp(iz_\nu^{(0)}|\alpha|)P_0(z_\nu^{(0)})}{\Delta_0'(\gamma, z_\nu^{(0)})}\right]$$

$$P_0(z) = c_1z^6 + c_2z^4 + c_3z^2 + c^4, \quad c_1 = -\frac{a_5}{\lambda^2 d_0^{(2)}}$$

$$c_2 = \frac{\gamma^2 d_0^{(8)}}{d_0^{(2)}}, \quad c_3 = \frac{d_0^{(6)} - a_3'}{d_0^{(2)}}, \quad c_4 = \frac{d_0^{(4)}}{d_0^{(2)}}$$

$$\Delta_k'(\gamma, z) = \frac{\partial}{\partial z}\Delta_k(\gamma, z)$$

Here, $z_\nu^{(k)}$ and $x_\nu^{(k)}$ are the complex and real roots of the polynomial (1.4), respectively, and $T = 2\pi/n$.

In the case of multiple roots the solution does not yield propagating waves since, when this is so, the group velocities are equal to zero. These values of the frequency parameter may be called resonance values /8/.

2. The amplitudes of the displacements and the electric field potential in the shell can be found in terms of the fundamental solution using well-known relationships /6/.

We note that the displacements and electric field potential in a shell which are brought about by the action of a periodic system of concentrated forces which vary harmonically with time represent the superposition of waves of a different kind. The monochromatic waves which propagate from the source along α correspond to the real roots of the equation $\Delta_k(\gamma, z) = 0$, while the inhomogeneous waves which decay exponentially along α as they become more remote from the source correspond to the complex roots. Each pair of complex roots $z_\nu^{(k)}, -z_\nu^{(k)}$ defines a standing wave along α while each pair of purely imaginary roots defines a standing wave along β with an exponentially decaying amplitude along α. Hence, the band of periods $0 \leqslant \beta \leqslant T$, $-\infty < \alpha < \infty$ acts as a waveguide **through** which energy is transported from the source of the perturbation to infinity.

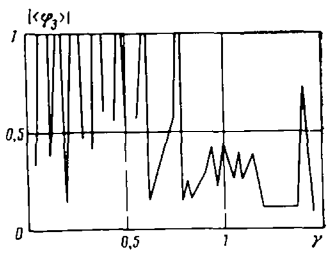

Fig.5

The amplitudes of the relative displacements u_{lm} $(m = 1, 2, 3)$ and the electric field potential (apart from an arbitrary constant) from the action of the external concentrated forces $P_l e^{-i\omega t}$ $(l = 1, 2, 3)$, $P_4 = 0$ have the form

$$\langle u_{lm}(\alpha, \beta)\rangle = \sum_{\nu=1}^{4} \frac{i \exp(i\alpha z_\nu^{(0)})\, P_0(z_\nu^{(0)})\, B_0^{lm}(z_\nu^{(0)})}{\Delta_0'(\gamma, z_\nu^{(0)})} + \tag{2.1}$$

$$\sum_{k=1}^{\infty} \left[2\cos(kn\beta) \sum_{\nu=1}^{\nu_*} \frac{\exp(iknx_\nu^{(k)}\alpha)\, B_k^{lm}(x_\nu^{(k)})}{\Delta_k'(\gamma, x_\nu^{(k)})} + \right.$$

$$\left. 2\,\mathrm{Re} \sum_{\nu=\nu_*+1}^{5} \frac{\exp(ikn\zeta_\nu^{(k)})\, B_k^{lm}(z_\nu^{(k)})}{\Delta_k'(\gamma, z_\nu^{(k)})} \right]$$

$$\zeta_\nu^{(k)} = z_\nu^{(k)}\alpha + \beta, \ \mathrm{Im}\, z_\nu^{(k)} > 0, \ x_\nu^{(k)} > 0$$

Here, B_k^{lm} are polynomials with coefficients which depend on the material, the shell parameters and also on the frequency parameter γ.

As an example, let us consider the vibrations of a cylindrical shell made of PZT-5 piezoceramic with $\lambda = 30$ and $n = 1$ under the action of forces which are concentrated at the point $\alpha_0 = 0, \beta_0 = 0$.

The amplitude-frequency characteristic of the longitudinal displacement u_1 (when $P_1 \neq 0$, $P_2 = P_3 = 0$), calculated using formula (2.1) for a piezoceramic shell (the solid line) and for a non-electric shell (the broken line) is shown in Fig.4. The calculations for the non-electric shell was carried out using (2.1) with $e_{15} = e_{33} = e_{31} = 0$.

The amplitude-frequency characteristic of the electric field potential φ_3 (under the action of P_3) is shown in Fig.5 for the same shell parameters.

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